LOCAL WEIGHTED GAUSSIAN CURVATURE FOR IMAGE PROCESSING

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ABSTRACT

We present a variational model with local weighted Gaussian curvature as regularizer. We show its convexity for an area-weight function and provide a closed-form solution for this case. The corresponding regularization coefficient has a theoretical bound. Moreover, we prove that the model is convex for a wide range of weight functions and show that it can be efficiently solved using splitting techniques. Finally, we demonstrate several applications of the model in image denoising, smoothing, texture decomposition, image sharpening, and regularization-coefficient optimization.

Index Terms— Gaussian curvature, regularization, convex model, variational form, image processing

1. INTRODUCTION

Reconstructing a signal from discrete samples, such as image pixels or a point cloud, is a fundamental task. However, since both the topology and the metric on the samples are missing, it is not clear what the true signal should be, especially in regions devoid of samples. Conceptually, there are two approaches to recovering the signal: interpolation (find missing data) and model fitting (reduce error). Both approaches require predefined basis functions that ideally reflect geometric properties of the signal, such as connectivity, smoothness, sparsity, or curvature. These implicitly assumed properties constitute the prior knowledge about the signal. Their imposition may render the reconstruction problem well-posed. Frequently used priors include sparsity in the spatial and/or frequency domain, total variation (TV), mean curvature (MC) [1, 2, 3], and Gaussian curvature (GC) [4, 5, 2, 6, 7].

Variational methods have been successfully used in image restoration [8, 9, 10, 2, 11], segmentation [12, 13], and inpainting [14]. Here, we show how to impose weighted Gaussian curvature (WGC) priors in a variational framework.

1.1. Variational Framework

Let \( S = \{ s_i(\vec{x}) : i = 1 \ldots N \} \) be the samples with spatial positions \( \vec{x} = (x, y)^T \). We aim at recovering an image \( U(\vec{x}) \) such that

\[
\min_{U \in F} E(U) = \int_{\vec{x} \in \Omega} \Phi_1(U, S) \, d\vec{x} + \lambda \int_{\vec{x} \in \Omega} \Phi_2(U) \, d\vec{x}.
\]

(1)

where \( \Phi_1 \) is a data-fitting (loss) functional between the recovered image \( U \) and samples \( S \). \( \Phi_2 \) is a regularization functional on \( U \). The parameter \( \lambda \) is a scalar regularization coefficient. \( \Omega \) is the image domain, and \( F \) is a suitable function space for \( U \).

Frequently, \( \Phi_1 \) is a distance metric, such as the Euclidean distance, Mahalanobis distance, Hausdorff distance, or \( L_2 \) distance. The choice of distance metric depends on how the data were obtained, the noise distribution and magnitude, the targeted reconstruction error, and the desired computational efficiency. Common choices are the \( L_2 \) distance to filter Gaussian noise or the \( L_1 \) distance to filter outliers.

\( \Phi_2 \) has to be designed with several goals in mind: 1) it should be efficient to compute; 2) it should have a mathematical meaning; 3) it should generate satisfactory results; 4) it should be easily adopted into different models. Even though there are many well-known regularization terms, such as Tikhonov, the \( \ell_2 \) norm of the gradient, TV, MC, total curvature (TC) [13], etc., none of them fulfills all of these characteristics. We show that WGC regularization has all of the above features.

Recently, curvature regularization has been adopted in variational frameworks for various image-processing problems, including inpainting [14], smoothing [15, 13], and segmentation [13, 16].

1.2. Gaussian Curvature

Let \( \vec{\Psi} = (\vec{x}, U(\vec{x})) \) be the image surface. We then have the first and second fundamental form:

\[
F = \begin{pmatrix}
1 + U_x^2, & U_x U_y \\
U_x U_y, & 1 + U_y^2
\end{pmatrix}
\]

(2)

\[
D = \begin{pmatrix}
\vec{\Psi}_{xx} \cdot \vec{n}, & \vec{\Psi}_{xy} \cdot \vec{n} \\
\vec{\Psi}_{yx} \cdot \vec{n}, & \vec{\Psi}_{yy} \cdot \vec{n}
\end{pmatrix},
\]

(3)

where subscripts denote differentiation with respect to the corresponding variable. The normal vector is given by \( \vec{n} = \).
\[
\left(\frac{-U_{x} - U_{y}}{\sqrt{1 + U_{x}^2 + U_{y}^2}} \right). 
\]
Gaussian curvature (GC) is defined as:
\[
G(U(\vec{x})) = \frac{\det(\mathbf{D})}{\det(\mathbf{F})} = \frac{U_{xx}U_{yy} - U_{xy}^2}{(1 + U_{x}^2 + U_{y}^2)^2}. 
\]
From the right-hand side of Eq. 4, we have:
\[
G(U(\vec{x})) = \left(U_{xx}U_{yy} - U_{xy}^2\right)/(1 + 2\|\nabla U\|^2_2 + \|\nabla U\|^2_2), 
\]
which means that \(U_{xx}U_{yy} - U_{xy}^2\), is a good approximation to \(G\) when \(\|\nabla U\|^2_2\) is small. This inspires our construction of an area-weighted Gaussian curvature regularizer for variational problems.

Previously, GC has been used in several diffusion-based models [4, 2, 6, 7], which are generally based on the geometric flow [4]
\[
\frac{\partial}{\partial t} U = \nabla \cdot (\phi(G)\nabla U) 
\]
with initial condition \(U_0 = S\) and proper boundary conditions. The function \(\phi\) is monotonic. This anisotropic diffusion process is similar to the Perona-Malik model [17]. Edge-indicator weights can be used to preserve edges during flow evolution [7]. A comparison of MC, GC, and TV has been done in Refs. [6, 2]. The model we present here is not diffusion-based.

1.3. Motivation and Contributions

GC is an intrinsic property of the surface and is independent of how the surface is embedded in external coordinates. Moreover, surfaces with zero GC can be isometrically mapped onto a plane without distortion. Minimizing GC can hence be seen as making the image surface As Planar As Possible (APAP). The Ricci flow drives the surface toward constant GC by evolving its Riemann metric [18].

Total GC, however, is related to the surface’s topology through the Gauss-Bonnet theorem:

**Theorem 1** \(\int_{\Omega} G d\vec{\Psi} + \int_{\partial\Omega} G_b d\vec{b} = 2\pi \chi(\vec{\Psi})\),

where \(G_b\) is the boundary curvature, \(d\vec{b}\) a length element, and \(\chi\) the Euler characteristic of \(\vec{\Psi}\). Because of this dependence, we minimize WGC instead of total GC.

A second reason is that the WGC model is more general, since different weight functions can be adopted. The resulting model is convex over a wide range of weight functions.

The Euler-Lagrange equation relates the variational framework (Eq. 1) to diffusion models (Eq. 6). For diffusion flows, however, a CFL stability condition has to be satisfied at every iteration, limiting computational performance especially for large images or videos. Convex models can be solved without any CFL limit using solvers such as Primal/Dual methods [19] or split-Bregman methods [20].

**Our contributions here are:**

1) We propose a new variational model with WGC regularization that is not based on anisotropic diffusion.

2) We prove that our model is convex for a wide range weight functions and present a closed-form solution for an area-weight function.

3) We provide a theoretical bound for the regularization coefficient and analyze a low-rank approximation to our model.

4) We derive an orthogonal basis that enables multi-resolution analysis for images of the same size.

5) We demonstrate several applications of WGC priors in image denoising, sharpening, and cartoon/texture decomposition.

2. WEIGHTED GAUSSIAN CURVATURE

We take \(\Phi_1 = \frac{1}{2}(U - S)^2\) and \(\Phi_2 = G(U)\theta(\vec{x})\), where \(\theta(\vec{x})\) is a weight function. We further take \(F_s\) to be the \(l^2\) space. Then, our model is defined as:
\[
\min_{U \in \ell^2} E(U) = \int_{\vec{x} \in \Omega} \frac{1}{2}(U - S)^2 d\vec{x} + \lambda \int_{\vec{x} \in \Omega} G(U)\theta d\vec{\Psi}. 
\]

2.1. Closed-Form Solution for Area Weights

In principle, the weight function \(\theta\) can be chosen arbitrarily. Motivated by Eq. 4, we choose \(\theta(\vec{x}) = (1 + U_{x}^2 + U_{y}^2)^{\frac{3}{2}}\), which is related to the surface area element \(d\Psi = d\vec{x} / \sqrt{1 + U_{x}^2 + U_{y}^2}\). The resulting WGC hence becomes an area-weighted GC. This weight is also the determinant of the Hessian matrix, which is commonly used for point or line detection, and is also related to tensor diffusion. We hence have:
\[
G(U)\theta(\vec{x}) d\vec{\Psi} = \det(\mathbf{D}) \det(\mathbf{F}) d\vec{x} = (U_{xx}U_{yy} - U_{xy}^2) d\vec{x}. 
\]

The resulting energy functional is:
\[
E(U) = \int_{\Omega} \frac{1}{2}(U - S)^2 d\vec{x} + \lambda \int_{\Omega} (U_{xx}U_{yy} - U_{xy}^2) d\vec{x}. 
\]

It can be rewritten in discrete form as:
\[
E(\hat{U}) = \frac{1}{2}(\hat{U} - \hat{S})^T (\hat{U} - \hat{S}) + \lambda \hat{U}^T W \hat{U}, 
\]

where \(\hat{U}\) and \(\hat{S}\) are discrete forms of \(U\) and \(S\), respectively. \(W = A_{xx}A_{yy} - A_{xy}^2 A_{xy}\), where \(A_{..}\) is the matrix of central-difference approximations to the second derivatives with respect to the subscript variables.

This model has a closed-form solution, which is not possible for diffusion-based models [4, 2, 6, 7]:
\[
\frac{\partial}{\partial U} E(\hat{U}) = (\hat{U} - \hat{S}) + \lambda W_2 \hat{U} = 0 
\]
\[
\implies (I + \lambda W_2) \hat{U} = \hat{S}, 
\]
where \(W_2 = W^T + W\).
2.1.1. Computational Efficiency

It is worth noting that \( M = I + \lambda W^2 \) is independent of \( \hat{U} \), which means that the entire matrix can be pre-computed and reused for images of the same size, as in a video sequence. Moreover, \( W \) is a very sparse matrix, which renders solving the above equation efficient. Table 1 compares the runtime of our model with that of a TV model [8]. This efficiency also allows optimizing the regularization parameter \( \lambda \) using line search.

<table>
<thead>
<tr>
<th>Image Size</th>
<th>Our Model</th>
<th>TV(^1)</th>
<th>TV(^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>64 × 64</td>
<td>0.02496</td>
<td>2.187</td>
<td>0.9409</td>
</tr>
<tr>
<td>128 × 128</td>
<td>0.1314</td>
<td>2.354</td>
<td>2.892</td>
</tr>
<tr>
<td>256 × 256</td>
<td>0.5963</td>
<td>6.041</td>
<td>5.983</td>
</tr>
</tbody>
</table>

Table 1. Runtime in seconds on a 2 GHz Intel Core i7 using Matlab R2012b.

2.1.2. Convexity and Bound for \( \lambda \)

We provide a proof that \( W \) is positive-semidefinite (PSD), implying that our model is convex, if \( \lambda \) is bounded. Let

\[
P = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 1 \\
1 & 0 & \cdots & \cdots & 0
\end{pmatrix}
\]

be a circulant square matrix of size \( mn \times mn \) (\( m \times n \) are the image dimensions). It is then clear that \( P^{mn} = I \) and \( A_{xx} = -2I + P + P^{-1} \). Similarly, \( A_{yy} = -2I + P^m + P^{-m} \) and \( A_{xy} = \frac{1}{4}(P^{m+1} + P^{-m-1} - P^{m-1} - P^{-m+1}) \). Therefore, \( W \) is PSD with spectral radius \( \rho(W) \leq 16 \). This implies that \( M \) is PSD if \( \lambda > -\frac{1}{\rho(W)} \).

2.1.3. Multi-Scale Analysis and Low-Rank Approximation

The eigenvectors \( v_i \) of \( W \) provide a complete orthogonal basis for all images of size \( m \times n \). The resulting basis is reminiscent of certain wavelet filters. Some example basis functions are shown in Fig. 1. They also provide a novel way to solve Eq. 12 in the sense of a low-rank approximation.

Let \( Wv_i = \gamma_i v_i, i = 1, \ldots, (mn) \), where \( \gamma_i \) are the corresponding eigenvalues. Then, \( \hat{S} = \sum_{i=1}^{mn} \beta_i v_i \), and \( \hat{U} = \sum_{i=1}^{mn} \alpha_i v_i \). Due to orthogonality, \( \alpha_i = \frac{\beta_i}{1+2\gamma_i} \). We solve this linear system for \( i < K \), where \( K \) is a pre-defined rank-approximation order.

2.2. General Weights with Edge Preservation

In the general case when \( \theta(\bar{x}) = \omega(\|\nabla \Psi\|) \), Eq. 7 can be rewritten in discrete form as:

\[
E(\hat{U}) = \frac{1}{2}(\hat{U} - \hat{S})^T(\hat{U} - \hat{S}) + \lambda \hat{U}^T W_\omega \hat{U},
\]

where \( W_\omega \) is a diagonal matrix corresponding to the weight function \( \omega \). More specifically, \( W_\omega(i,i) = \frac{\omega(\|\nabla \Psi\|)}{(1+\gamma_i)^2} \).

Then:

\[
\frac{\partial}{\partial \hat{U}} E(\hat{U}) \approx (\hat{U} - \hat{S}) + \lambda W_\omega \hat{U} = 0 \quad \text{(14)}
\]

\[
\Rightarrow (I + \lambda W_\omega)\hat{U} = \hat{S} \quad \text{(15)}
\]

This model has no closed-form solution. However, it can be efficiently solved using the split weighted Gaussian curvature (SWGC) algorithm given below.

Algorithm 1 SWGC: Split Weighted Gaussian Curvature

Require: \( \hat{S}, \lambda \)
1: compute \( W_2, W_\omega(\hat{S}) \), set \( \hat{U}_0 = \hat{S}, k = 0 \)
2: while max\{\|\hat{U}_{k+1} - \hat{U}_k\|\} < tol do
3: compute \( W_\omega(\|\nabla \Psi\|) \)
4: compute \( \hat{U}_{k+1} \) from Eq. 15
5: \( k = k + 1 \)
6: end while

Ensure: \( \hat{U} \)

2.2.1. Convexity and Bound for \( \lambda \)

\( W_\omega \) is PSD for a wide range of functions \( \omega \). Therefore, \( WW_\omega \) is PSD and \( \lambda \) is bounded as \( \lambda > -\frac{1}{\rho(WW_\omega)} \). It is straightforward to get a bound for \( \omega \). For example, when \( \omega \) is the identity function, \( \rho(W_\omega) \leq 1 \).
3. APPLICATIONS

We demonstrate the application of WGC in image smoothing, sharpening, cartoon/texture decomposition, image denoising, and regularization-coefficient optimization.

3.1. Image Smoothing and Sharpening

Figure 2 shows the results of image smoothing and sharpening using our WGC model with the area-weight function and with different parameters (sharpening with negative \( \lambda > -\frac{\epsilon}{\max(W)} \)). A line profile is compared with TV\(^3\) in Fig. 3.

In Fig. 4, image smoothing is shown with edge-preserving weights (\( \omega \) is the identity function); a detail patch is shown in the row below. In practice, three to four iterations of SWGC are enough.

3.2. Cartoon/Texture Decomposition and Denoising

We compare the area-weighted GC model with TV regularization for cartoon/texture decomposition and image denoising. The result is shown in Fig. 5 for \( \epsilon = 1.5 \) in TV and \( \lambda = 30 \) in WGC.

For denoising, the image is corrupted with additive Gaussian noise of magnitude \( \sigma = 10 \). Denoising results by TV\(^3\) and WGC (area weight) with \( \lambda = 0.48 \) are shown in Fig. 5. The final mean-square errors for TV and WGC are 70.77 and 56.54, respectively.

\(^3\)TV uses \( \epsilon = 1 \) and max iteration = 80

3.3. Regularization-Coefficient Optimization

To the best of our knowledge, searching for the optimal regularization coefficient \( \lambda \) is hard in general. The computational efficiency of our model, however, allows the use of line search to optimize the regularization coefficient in Eq. 7 by solving Eq. 12. We did this for the image denoising experiment above.

4. CONCLUSION AND FURTHER WORK

We have presented weighted Gaussian curvature regularization in a variational framework. The resulting model is convex over a wide range of weight functions and has a closed-form solution for the special case of area weights. We have shown a bound on \( \lambda \) and presented an efficient algorithm to numerically solve the model when no closed-form solution is available. We have demonstrated the proposed model in several applications ranging from image smoothing to sharpening, denoising, and cartoon/texture decomposition.

Weighted Gaussian curvature can be further extended to 3D images and to point-cloud surfaces [21].
5. REFERENCES


