Drawing an elephant with four complex parameters

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We define four complex numbers representing the parameters needed to specify an elephantine shape. The real and imaginary parts of these complex numbers are the coefficients of a Fourier coordinate expansion, a powerful tool for reducing the data required to define shapes.

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A turning point in Freeman Dyson's life occurred during a meeting in the Spring of 1953 when Enrico Fermi criticized the complexity of Dyson's model by quoting Johnny von Neumann:[1] "With four parameters I can fit an elephant, and with five I can make him wiggle his trunk." Since then it has become a well-known saying among physicists, but nobody has successfully implemented it.

To parametrize an elephant, we note that its perimeter can be described as a set of points $(x(t),y(t))$, where t is a parameter that can be interpreted as the elapsed time while going along the path of the contour. If the speed is uniform, t becomes the arc length. We expand x and y separately[2] as a Fourier series

$$x(t)=\sum_{k=0}^{\infty}(A_k^x\cos(kt)+B_k^x\sin(kt)), \tag{1}$$

$$y(t)=\sum_{k=0}^{\infty}(A_k^y\cos(kt)+B_k^y\sin(kt)), \tag{2}$$

where A_k^x, B_k^x, A_k^y, and B_k^y are the expansion coefficients. The lower indices k apply to the kth term in the expansion, and the upper indices denote the x or y expansion, respectively.

Using this expansion of the x and y coordinates, we can analyze shapes by tracing the boundary and calculating the coefficients in the expansions (using standard methods from Fourier analysis). By truncating the expansion, the shape is smoothed. Truncation leads to a huge reduction in the information necessary to express a certain shape compared to a pixelated image, for example. Székely *et al.*[3] used this approach to segment magnetic resonance imaging data. A similar approach was used to analyze the shapes of red blood cells,[4] with a spherical harmonics expansion serving as a 3D generalization of the Fourier coordinate expansion.

The coefficients represent the best fit to the given shape in the following sense. The $k=0$ component corresponds to the center of mass of the perimeter. The $k=1$ component corresponds to the best fit ellipse. The higher order components trace out elliptical corrections analogous to Ptolemy's epicycles.[5] Visualization of the corresponding ellipses can be found at Ref. 6.

We now use this tool to fit an elephant with four parameters. Wei[7] tried this task in 1975 using a least-squares Fourier sine series but required about 30 terms. By analyzing the picture in Fig. 1(a) and eliminating components with amplitudes less than 10% of the maximum amplitude, we obtained an approximate spectrum. The remaining amplitudes were

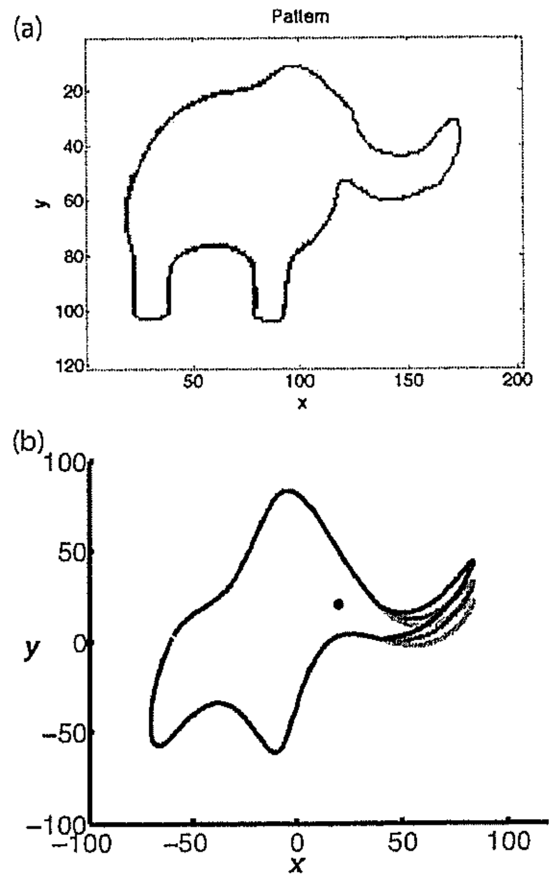

Fig. 1. (a) Outline of an elephant. (b) Three snapshots of the wiggling trunk.

Table I. The five complex parameters $p_1,\ldots,p_5$ that encode the elephant including its wiggling trunk.

Parameter	Real part	Imaginary part
$p_1=50-30i$	$B_1^x=50$	$B_1^y=-30$
$p_2=18+8i$	$B_2^x=18$	$B_2^y=8$
$p_3=12-10i$	$A_3^x=12$	$B_3^y=-10$
$p_4=-14-60i$	$A_5^x=-14$	$A_1^y=-60$
$p_5=40+20i$	Wiggle coeff.=40	$x_{eye}=y_{eye}=20$

slightly modified to improve the aesthetics of the final image. By incorporating these coefficients into complex numbers, we have the equivalent of an elephant contour coded in a set of four complex parameters (see Fig. 1(b)).

The real part of the fifth parameter is the "wiggle parameter," which determines the x-value where the trunk is attached to the body (see the video in Ref. 8). Its imaginary part is used to make the shape more animal-like by fixing the coordinates for the elephant's eye. All the parameters are specified in Table I.

The resulting shape is schematic and cartoonlike but is still recognizable as an elephant. Although the use of the Fourier coordinate expansion is not new,[2,3] our approach clearly demonstrates its usefulness in reducing the number of parameters needed to describe a two-dimensional contour. In the special case of fitting an elephant, it is even possible to lower it to four complex parameters and therein implement a well-known saying.

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[2]F. P. Kuhl and C. R. Giardina, "Elliptic Fourier features of a closed contour," Comput. Graph. Image Process. 18, 236–258 (1982).

[3]G. Székely, A. Kelemen, C. Brechbühler, and G. Gerig, "Segmentation of 2D and 3D objects from MRI volume data using constrained elastic deformations of flexible Fourier contour and surface models," Med. Image Anal. 1(1), 19–34 (1996).

[4]K. Khairy and J. Howard, "Spherical harmonics-based parametric deconvolution of 3D surface images using bending energy minimization," Med. Image Anal. 12(2), 217–227 (2008).

[5]The interactive Java applet written by Rosemary Kennett, ⟨physics.syr.edu/courses/java/demos/kennett/Epicycle/Epicycle.html⟩.

[6]Interactive Java applet of elliptic descriptors by F. Puente León, ⟨www.vms.ei.tum.de/lehre/vms/fourier/⟩.

[7]J. Wei, "Least square fitting of an elephant," CHEMTECH 5, 128–129 (1975).

[8]See supplementary material at http://dx.doi.org/10.1119/1.3254017 for the movie showing the wiggling trunk.